Robust Control System Design Synthesis with Observers

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A method of designing a robust controller including an observer for a multi-input/multi-output controlled system is presented. The well-known robustness of the optimal regulator is valid only when the regulator is constructed by full-state feedback, and this robustness is not guaranteed if an observer is introduced. In this study, we add a new output feedback loop to recover the robustness of the system with an observer and use a precompensator to improve the response characteristics. Singular-value analysis is applied to improve the robustness of the synthesized system. An example for an aircraft is calculated, and good simulated results are demonstrated.

I. Introduction

SYSTEM parameters of aircraft and missiles vary with their flight conditions. It is desirable that the stability and response characteristics of control systems are preserved under parameter variation. A control system with this property is said to be robust with respect to parameter variations. To synthesize a robust control system, methods using singular values have been developed for multi-input/multi-output (MIMO) systems. In this method, a measure of the robustness is the minimum singular value of the return difference matrix at the input or the output of the plant. The robustness of systems has been analyzed with the condition for robust stability and reduced sensitivity expressed in terms of singular values,[1,2] and the concept of classical stability margin has been extended to MIMO systems.[3,4] Numerical optimization techniques have also been studied to enhance the robustness.[5,6]

The regulator with optimal feedback is known to have the excellent robust stability and reduced sensitivity irrespective of the selected weighting matrices in the performance index.[7] Some of the present authors have demonstrated that the use of optimal feedback and the double perfect model following will further improve robustness.[8] This robustness of the optimal regulator is limited to the ideal case, though, where all the state variables can be detected and used for feedback. In practical systems such as aircraft and missiles, all the state variables are rarely measured. In these cases, though the state variables can be estimated by an observer or Kalman filter, the optimal feedback using the estimated state variables result in considerable degradation in the robustness compared with the aforementioned cases.[9] As the solutions to this problem, the linear quadratic Gaussian/loop transfer recovery (LQG/LTR) method of Doyle[9] and a design method using perfect regulation and perfect observation of Kimura[10] have been proposed. But since these methods are asymptotic, the gains for some signals become large, and saturation is probable in a practical system. On the other hand, some of the authors have proposed a method of designing robust control systems with an observer, where the return difference matrix is recovered to satisfy the circle condition.[11] In this method the closed-loop part is designed so that the return difference matrix coincides with that of the optimal regulator, and a precompensator part is added to equalize the response of the designed system to the desired one.

In this paper we propose a design method for robust control with observers using singular values, which follows the aforementioned idea and is readily applicable to MIMO systems. In particular, instead of requiring robustness as large as the optimal regulator, we recover the robustness by a numerical optimization technique of Newsom et al.[5] to the extent that the designer requires. In this method, the design requirement of robust stability and reduced sensitivity is designated as a specification on the minimum singular value, which is transformed to objective functions. The robust controller gain is then determined by a numerical calculation to minimize the feedback gain with constraints that these objective functions keep zero.

As a numerical example, a two-input/two-output control system for the lateral motion of an aircraft is designed by the proposed method. The resulting control system is simulated for a step input to show the expected performance.

II. Robustness

The sufficient conditions for a system to have robust stability and reduced sensitivity are expressed using singular values as follows.

A. Robust Stability

Assume that the nominal closed-loop system is stable. Then a sufficient condition for the system to remain robustly stable after the open-loop transfer function $G(s)$ is perturbed at the output to $L(s)G(s)$ is[1]

$$\underline{\sigma}[I+G(j\omega)] > \bar{\sigma}[L^{-1}(j\omega)-I], \qquad (0<\omega<\infty) \tag{1}$$

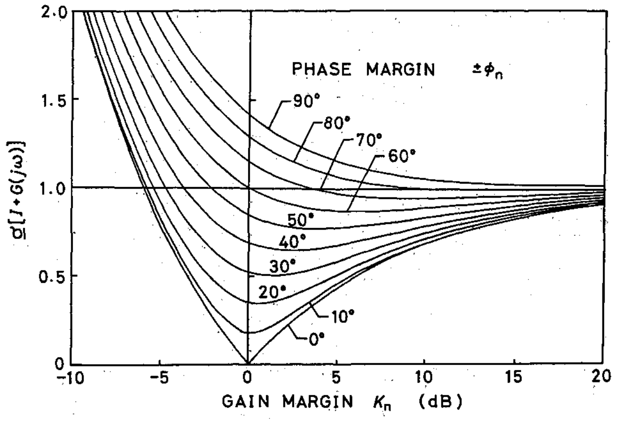

Fig. 1 Diagram for multiloop phase and gain margin evaluation.

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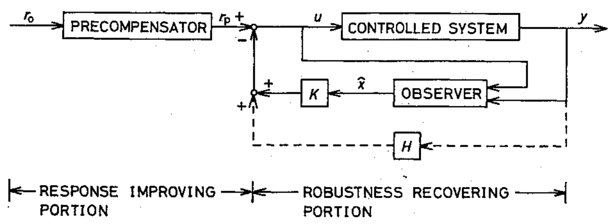

Fig. 2 System design configuration.

where $\bar{\sigma}$ and $\underline{\sigma}$ represent the maximum and the minimum singular values, respectively.

Consider the following diagonal perturbation as L in Eq. (1)

$$L = \text{diag}[k_n \exp(j\phi_n)], \qquad n = 1,2,\ldots,m \tag{2}$$

where m is the number of outputs. L reduces to the identity matrix for the nominal system. Substitution of Eq. (2) into L in the right-hand side of Eq. (1) yields

$$\underline{\sigma}[I + G(j\omega)] > \max_n \{[1-(1/K_n)]^2 + (2/K_n)(1-\cos\phi_n)\}^{1/2} \tag{3}$$

which corresponds to the case where the gains and phases vary simultaneously in all loops. From Eq. (3) the diagram for multiloop phase and gain margin evaluation in Fig. 1 is attained.[3,5] From this diagram the minimum value of $\underline{\sigma}[I + G(j\omega)]$ corresponding to the required gain margin (GM) or phase margin (PM) is obtained. For example, when both the gain and phase vary, the stability margin of the system for the minimum value of $\underline{\sigma}[I+G(j\omega)] = 0.6$ is -1.5 dB$<$GM$<+5.3$ dB and PM$=\pm 30$ deg, whereas when either the gain or the phase varies ($\phi_n = 0$ or $K_n = 1$), the stability margin of the system is -4.2 dB$<$GM$<+8$ dB or PM$=\pm 35$ deg.

B. Sensitivity Reduction

Sensitivity reduction here means that the effect of plant parameter variation appears smaller in the closed-loop system than in the nominally equivalent open-loop system. When the parameter variation is small and continuous, a sufficient condition for this is[1]

$$\underline{\sigma}[I + G(j\omega)] \geq 1, \qquad (\omega \leq \omega_0) \tag{4}$$

Though this condition is satisfied for the optimal regulator over the entire frequency range, it is considered sufficient for other regulators that this condition is satisfied within the frequency range up to the system bandwidth ω_0.

III. Robust Control System Design

A. Construction of the System

Consider a controlled system described by the following state equations:

$$\dot{x} = Ax + Bu \tag{5}$$

$$y = Cx \tag{6}$$

where $x \in R^n$, $u \in R^\ell$, $y \in R^m$ are vectors, A, B, C are constant matrices with appropriate dimensions, and the system is assumed to be controllable and observable. Since the output is y, as shown in Fig. 2, we use here an observer whose characteristics are described by

$$\dot{\hat{x}} = (A - FC)\hat{x} + Fy + Bu \tag{7}$$

where $\hat{x}$ is an estimate of x, and F is the optimal gain of the observer. The feedback gain K is taken as that of the optimal regulator.

The optimal regulator is generally used for the design of controllers for two reasons. The first is to optimize the response of the nominal plant, and the second is to realize the guaranteed robustness for reduction of sensitivity to plant parameter variation. When an observer is included, however, the return difference matrix at the output of the plant is different from that when the observer is not included. This explains that robustness is not guaranteed in this case even if the optimal feedback gain is used.

To recover the robustness and the optimal response in this case, we add a feedback H shown by the broken line in Fig. 2. The precompensator is used to obtain a desired response.

The open-loop transfer function at the output of this system is

$$G(s) = [C(sI-A)^{-1}B][K(sI - A + BK + FC)^{-1} \times (F - BH) + H] \tag{8}$$

The open-loop transfer function at the input of this system is given by an expression obtained by exchanging the two factors in Eq. (8) grouped by brackets.

B. Robustness Recovery Design

For robustness recovery, we determine H in Eq. (8) so that the return difference matrix at the output satisfies the robustness condition shown by Eqs. (3) and (4). It is also required that the synthesized closed-loop system is stable.

The characteristic equation of this system is described by the following two equations:

$$\det\,(sI - A + BK + BHC) = 0 \tag{9}$$

$$\det\,(sI - A + FC) = 0 \tag{10}$$

Since Eq. (10) is the equation of the observer, it is evidently stable. Therefore the closed-loop system is stable if Eq. (9) is stable. The conditions that Eq. (9) remains stable no matter how large H becomes are given as follows.[12]

Condition 1:

$$\text{rank } CB = m \qquad \text{and} \qquad m \leq \ell \tag{11}$$

Condition 2:

$$Q(s) = \begin{bmatrix} sI - A + BK & B \\ C & 0 \end{bmatrix}$$

satisfies

$$\text{rank } Q(s) = n + m \tag{12}$$

for any $s \in C^+$ where C^+ is the closed right-half complex plane divided at and including the imaginary axis.

If these conditions are satisfied, then H may be chosen as

$$H = gP \tag{13}$$

where P is a matrix and g is a scalar. Since the dimension of P is constrained by Eq. (11), we treat a case of the maximum dimension of output where $m = \ell$ and then P becomes square. In the case of $m < \ell$, one may apply our method for an m-dimensional subset of control loops with the remaining $(\ell - m)$ loops closed. The scalar g is selected as the minimum positive number, which reduces the following cumulative measures $J_1(g)$ and $J_2(g)$ to zero simultaneously to satisfy the robustness conditions of Eqs. (3) and (4).

Robust stability[5]

$$J_1(g) = \sum_i (\max\{0, [\sigma_D - \underline{\sigma}(j\omega_i, g)]\})^2 \qquad (0 < \omega < \infty) \tag{14}$$

Sensitivity reduction

$$J_2(g)=\sum_i(\max\{0,\ [1-\underline{\sigma}(j\omega_i,g)]\})^2 \quad (0<\omega<\omega_0) \tag{15}$$

where Σ means that the summation is calculated for many frequency points ω_i, which is chosen suitably within the specified frequency range. The cumulative measure of Eq. (14) aims to increase the minimum value up to the desired value $\underline{\sigma}_D$ over the entire frequency range. The $\underline{\sigma}_D$ is selected as the minimum value that gives the desired stability margin from the diagram for phase and gain margin evaluation in Fig. 1. On the other hand, Eq. (15) aims to make $\underline{\sigma}$ larger than unity within the specified frequency range where the bandwidth ω_0 is selected from the characteristics of the given system.

Now P is assumed to be a nonsingular diagonal matrix for simplicity in calculation. Though nondiagonal P can be used, it brings little benefit and a large increase of computing time. A g is obtained, which makes Eq. (14) and Eq. (15) zero simultaneously for the given values of $\underline{\sigma}_D$ and ω_0, and then H is determined. Large H is not preferable because it amplifies the effect of measurement noise since H is a direct feedback element of the system. Therefore minimization of H is desired. Now let us define the following norm:

$$\|H\|=\left[\sum_{i=1}^{m}|h_{ii}|^2\right]^{1/2}=\left[\sum_{i=1}^{m}g^2|p_{ii}|^2\right]^{1/2} \tag{16}$$

The flow of the computation is shown in Fig. 3. First, an initial value for P is given so that $(-PCB)$ is stable. An initial value of increment ΔP of P is also given previously. Then computation is started and an H, which makes Eqs. (14) and (15) zero, is obtained and $\|H\|$ for this H is computed from Eq. (16). Second, P is increased by ΔP and a new value of $|H|$ is computed by the same procedure for a new P. The new value of $\|H\|$ is compared with the old one, and the smaller value is kept. The same computation is repeated, altering ΔP automatically through a computer program based on the simplex method. When the variation of $\|H\|$, according to the variation of P, becomes smaller than a prescribed value, then the computation is terminated and H is adopted as the designed value.

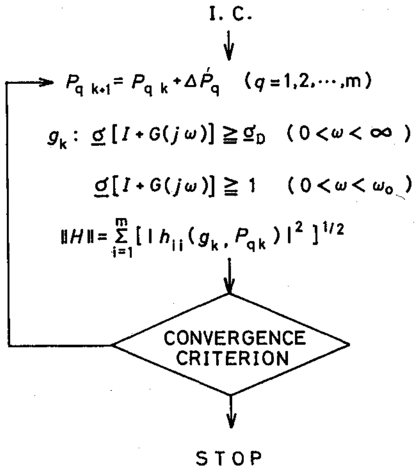

Fig. 3 Flow of the computation.

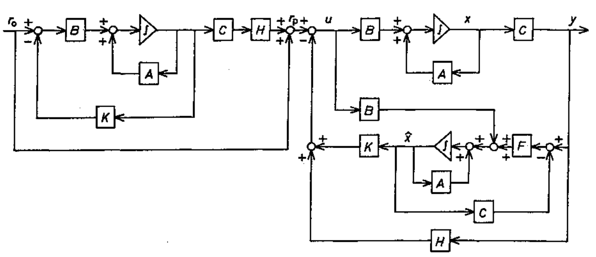

Fig. 4 Block diagram of the designed system.

C. Design to Improve the Response

The closed-loop transfer function of the optimal regulator

$$G_c(s)=C(sI-A+BK)^{-1}B \tag{17}$$

is different from that of the system having the additional loop of H,

$$G_c'(s)=C(sI-A+BK+BHC)^{-1}B \tag{18}$$

The resulting response characteristics of Eq. (18) are not necessarily desirable. Therefore a precompensator based on the model-following method was added to obtain a desirable response. A precompensator is designed so that the system behaves like that of the optimal regulator of Eq. (17). Let the transfer function of the precompensator be $G_f(s)$, then the overall transfer function from the input r_0 to the system output y is written as

$$C(sI-A+BK+BHC)^{-1}BG_f(s)=C(sI-A+BK)^{-1}B \times\{I+HC(sI-A+BK)^{-1}B\}^{-1}G_f(s) \tag{19}$$

In order to equalize Eq. (19) with Eq. (17), the transfer function of the precompensator is chosen as

$$G_f(s)=I+HC(sI-A+BK)^{-1}B \tag{20}$$

If other arbitrary response characteristics are desired, one may use prefilter[13] or extended perfect-model-following[14] methods that are applicable to a nonsquare case.

A block diagram of the designed system is shown in Fig. 4.

So far we treated the robustness at output. If the robustness at input is needed, the same procedure is applicable for a return difference matrix based on the previous open-loop transfer function at input. A method of evaluating the robustness at both input and output also can be considered.[15]

IV. Example of Application to a Flight Control System

The state equation for the lateral motion of an aircraft flying at a flight condition of 40,000 ft altitude and Mach = 0.80 is written as[16]

$$\dot{x}=\begin{bmatrix}-43.2 & 0.0416 & 0 & -1\\ 0 & 0 & 1 & 0\\ -3.05 & 0 & -0.465 & 0.388\\ 0.598 & 0 & -0.0318 & -0.15\end{bmatrix}x + \begin{bmatrix}0 & 0.00729\\ 0 & 0\\ 0.143 & 0.153\\ 0.00775 & -0.475\end{bmatrix}u \tag{21}$$

where

$$x=(\beta \quad \phi \quad p \quad r)^T, \qquad u=[\delta_a \quad \delta_r]^T$$

and

β = sideslip angle
p = roll rate
δ_a = aileron angle
ϕ = roll angle
r = yaw rate
δ_r = rudder angle

Assuming that p and r are measurable, we obtain the following output equation

$$y=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \tag{22}$$

We assume that an optimal regulator has the desired response characteristics when the weighting matrices in the performance index are chosen as Q = diag [100, 100] and R = diag [1, 1]. For these weighting matrices, the optimal gain is obtained as follows:

$$K=\begin{bmatrix} -0.463 & 0 & 7.25 & 2.37 \\ -0.134 & 0.00300 & 0.418 & -10.1 \end{bmatrix} \tag{23}$$

On the other hand, we use a Kalman filter-type observer and choose the covariance matrices for fictitious noise at input and output as the same values of Q and R. Then the observer gain F is obtained as

$$F=\begin{bmatrix} 0.0171 & 0.00689 & 1.34 & -0.787 \\ -0.102 & -0.117 & -0.787 & 4.56 \end{bmatrix}^T \tag{24}$$

Substituting the preceding system parameters into Eqs. (11) and (12), we see that the conditions 1 and 2 are satisfied, and therefore a stable H exists and the proposed robust control system can be designed.

Now we designate the following performance about the robustness of the designed system.

Robust stability

$$\underline{\sigma}_D = 0.90$$

This corresponds to

$$-5.6 \text{ dB} < \text{GM} < +20.0 \text{ dB}$$

$$\text{PM} = \pm 54 \text{ deg}$$

as obtained from Fig. 1, and these stability margins are adequate for servo systems.

Sensitivity reduction

$$\omega_0 = 10 \text{ rad/s}$$

This value is considered to be adequate for the bandwidth of the lateral motion of aircrafts.

The numerical calculation is performed according to the flow in Fig. 3. The simplex method was used as the minimization procedure. Using the initial vertex value

$$P(1,1)=1, \qquad P(2,2)=-1$$

we obtain the final values of P and g as

$$P=\begin{bmatrix} 1.244 & 0 \\ 0 & -0.857 \end{bmatrix}$$

$$g = 11.31$$

from which H for the minimum norm becomes

$$H = gP = \begin{bmatrix} 14.07 & 0 \\ 0 & -9.47 \end{bmatrix} \tag{25}$$

To confirm that the designed system with this H improves robustness, the minimum singular value vs frequency plot is shown by a dot-dash line in Fig. 5. Those for the optimal regulator (solid line) and for the optimal regulator with observer (dotted line) are also shown in the same figure. The optimal regulator has the guaranteed stability margins

$$-6 \text{ db} < \text{GM} < +\infty \text{ dB}$$

$$\text{PM} = \pm 60 \text{ deg}$$

The stability margins for the optimal regulator with observer are

$$-5.4 \text{ dB} < \text{GM} < 17.7 \text{ dB}$$

$$\text{PM} = \pm 52 \text{ deg}$$

since the minimum value of σ is 0.87. As for the sensitivity of the optimal regulator with observer, $\underline{\sigma}$ is less than 1 within $\omega > 0.9$ rad/s, which does not satisfy the design specification. On the other hand, the designed system constructed with the above H has $\underline{\sigma} \geq 1$ up to the frequency range of $\omega_0 = 10$ rad/s, satisfying the design specification on sensitivity. It has also a minimum value of 0.98, which fully satisfies the design specification on stability.

The precompensator that results in the same response as the optimal regulator is designed using Eq. (20), and the system is constructed as shown in Fig. 4.

V. Simulation

Simulated responses for the following systems are shown in Figs. 6 and 7 as nominal: 1) an optimal regulator for the plant of Eq. (21) with K of Eq. (23); 2) an optimal regulator with observer for the plant of Eqs. (21), (22), and (9) with K of Eq. (23) and F of Eq. (24). (This system is from r_p to y without H in Fig. 2); and 3) the proposed control system in Fig. 4 with K, F, and H in Eqs. (23–25).

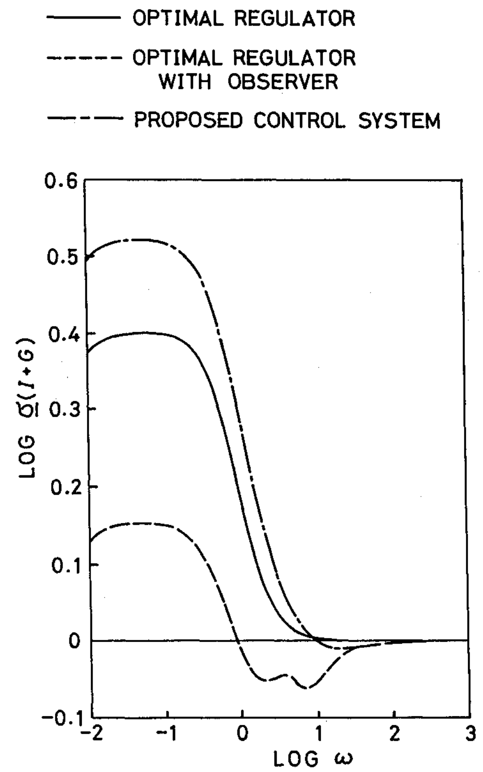

Fig. 5 Singular value plot of the designed system.

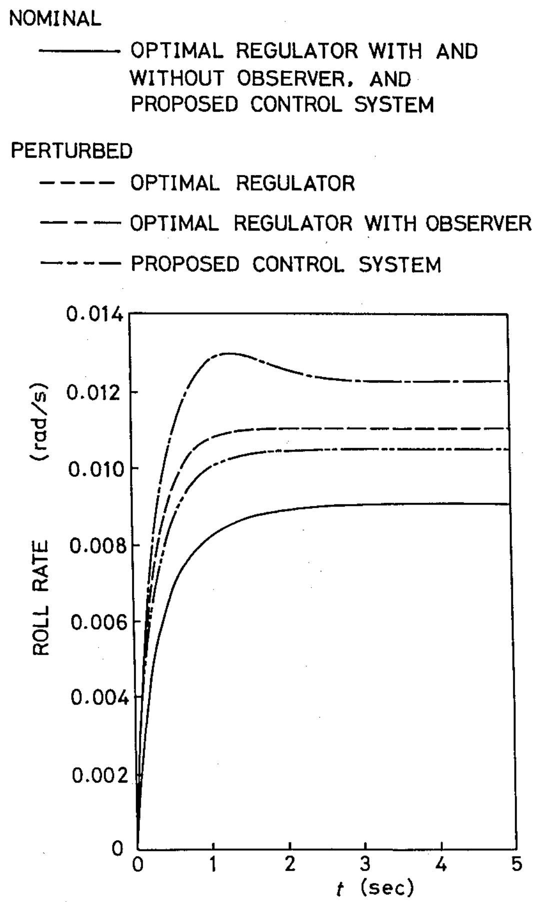

Fig. 6 Roll-rate step response.

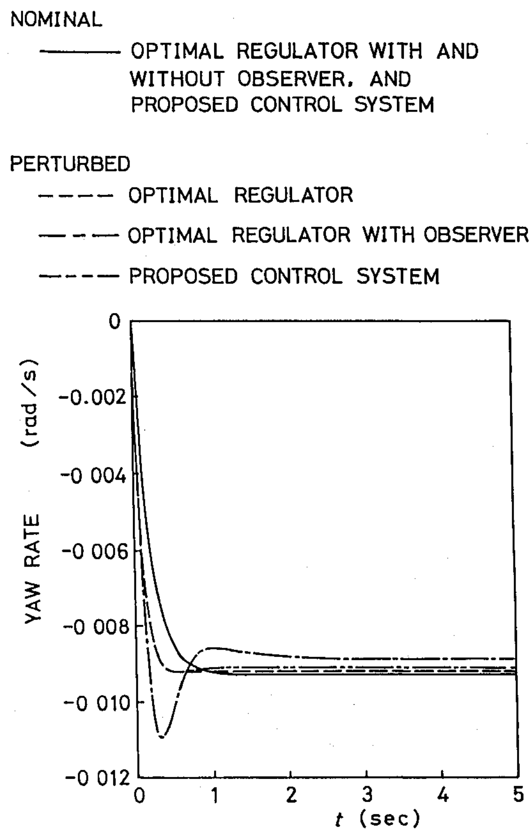

Fig. 7 Yaw-rate step response.

A step input

$$r_p = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

was applied to systems 1 and 2 and a step input

$$r_0 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

was applied to system 3. All initial conditions were zero. The output p (roll rate) and r (yaw rate) of nominal systems are shown by solid lines in Fig. 6 and Fig. 7, respectively. Since systems 1–3 are nominally equivalent to each other, their responses perfectly coincide as seen in these figures.

To confirm the robustness, a parameter variation is considered. When the flight condition of the aircraft is varied to 20,000 ft altitude and Mach = 0.80, the parameters are varied as follows:

$$A = \begin{bmatrix} -99.4 & 0.0388 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -4.12 & 0 & -0.974 & 0.292 \\ 1.62 & 0 & -0.0157 & -0.232 \end{bmatrix} \tag{26}$$

$$B = \begin{bmatrix} 0 & 0.0124 \\ 0 & 0 \\ 0.310 & 0.183 \\ 0.0127 & -0.922 \end{bmatrix} \tag{27}$$

Simulated responses of the system with a perturbed plant for the previous inputs are shown in Figs. 6 and 7 by broken lines for system 1 by one dotted, broken line for system 2 and by two dotted, broken lines for system 3. The responses of the optimal regulator with the observer (system 2) show large overshoot and are deviated from the nominal responses (solid lines) considerably. The responses of the present designed system (system 3) are close to those of the optimal regulator without the observer (system 1). Particularly, the response of the roll rate is rather close to the nominal response (solid line).

As shown in this example, compared with the optimal regulator without an observer, the system designed by the proposed method has more or equally reduced effect for parameter variation and has a sufficient stability margin, and therefore its robustness is considered to be recovered.

VI. Conclusions

A method of designing a feedback controller that improves the robustness of multi-input/multi-output systems, including an observer, has been demonstrated. Although robustness improvement generally results in degraded response, this problem has been solved with a two-stage construction (two-freedom design) where robustness is recovered by direct output feedback, and the response is improved by precompensation. The numerical example demonstrated the effectiveness of the proposed design method.

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